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Chris Bernhardt
Fairfield University, cbernhardt@fairfield.edu

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VERTEX MAPS FOR TREES: ALGEBRA AND PERIODS OF PERIODIC ORBITS

CHRIS BERNHARDT

Department of Mathematics and Computer Science
Fairfield University
Fairfield, CT 06824, USA

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ABSTRACT. Let T be a tree with n vertices. Let $f : T \to T$ be continuous and suppose that the n vertices form a periodic orbit under f. The combinatorial information that comes from possible permutations of the vertices gives rise to an irreducible representation of S_n. Using the algebraic information it is shown that f must have periodic orbits of certain periods. Finally, a family of maps is defined which shows that the result about periods is best possible if $n = 2^k + 2^l$ for $k, l \geq 0$.

1. **Introduction.** Sharkovskii first proved his celebrated theorem in [12] in 1962. Since then there have been various proofs given of this result. Probably the simplest method of proof is a directed graph proof first given by Block, Guckenheimer, Misiurewicz and Young in [10] and Ho and Morris in [11]. A variation on these ideas is given in [7]. In this proof each periodic orbit on the interval is associated with a permutation that comes from the order in which the points in the orbit are permuted. It is then shown how a matrix can be assigned to each permutation. The proof of Sharkovsky's theorem then follows fairly simply from three results: the map from the group of permutations to matrices is a homomorphism, if a permutation does not contain a 1-cycle then the trace of the associated matrix is -1, and if the matrix associated to a cyclic permutation has three or more non-zero entries along the major diagonal then the underlying map must have periodic orbits of all periods. This paper looks at how those results can be extended from the interval to trees. Before proceeding further we need to give a precise definition of what we will mean by tree, vertex and edge.

A space homeomorphic to the closed interval $[0, 1]$ is called an *edge*. The endpoints of an edge are called *vertices*. A *tree* is a finite union of edges that is contractible to a point and has the property that the intersection of two distinct edges is either empty or consists of one vertex.

It should be noted that this definition of vertex comes from the combinatorial viewpoint, where it is possible for the removal of a vertex to decompose the tree into two connected components, rather than from the topological approach, where removal of a vertex cannot decompose the tree into exactly two disconnected components. Thus, for example, we will consider $[0, 1] \cup [1, 2]$ to be a tree with two edges, $[0, 1]$ and $[1, 2]$, and with three vertices, 0, 1 and 2.

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In what follows we will be considering maps under which the vertices form a periodic orbit. Our approach does include periodic orbits on the interval, since we can consider the tree to be the convex hull of the periodic orbit and the periodic orbit to form the vertices. However, requiring the vertices to form a periodic orbit is a very special case of the theory of maps on trees. (It should be noted that there have been a number of papers that have studied maps on trees without this restriction. Baldwin began this study by considering certain types of trees (see [6]) and it has been continued by work of Alseda working in conjunction with Guaschi, Juher, Llibre, Los, Misiurewicz, Manosas, Mumbru and Ye (see [1], [2], [3] and [4]).)

As in [7] it will be shown that we can associate a matrix to each permutation. Again, it is shown that this construction gives a representation of the symmetric group. In fact we prove a little more, showing that these representations are irreducible. We also show that if the permutation contains no 1-cycles then the trace of the associated matrix is -1. However, we show that it is not true that if the major diagonal of the associated matrix has three non-zero entries that the map must have periodic orbits of all periods.

From the results stated above we can prove the following theorem.

Theorem 1. *Let T be a tree with n vertices that form a periodic orbit under the map f.*

1) If n is not a divisor of 2^k then f has a periodic point with period 2^k.

2) If $n = 2^k m$, where $m > 1$ is odd and $k \geq 0$, then f has a periodic point with period $2^k l$ for any $l \geq m$.

3) If n is odd then f has a periodic point with period $n - 1$

We conclude the paper with a family of trees and maps that has exactly the periods given by Theorem 1 when the binary expansion of n contains two or fewer 1s.

2. **Basic Ideas.** Let T be a tree with n vertices. We will label the vertices with the integers from 1 to n. The tree has $n - 1$ edges that can be described by their endpoints. If k and l are vertices and there is an edge between them we will denote the *positively oriented edge* from k to l as $[kl]$. For each edge we choose an orientation and we will label the positively oriented edges as $E_1, \ldots, E_{n-1}$. If $E_p = [kl]$ then $[lk]$ describes the same edge, but with the opposite (negative) orientation. We will denote $[lk]$ by $-E_p$. We will call k the *first* vertex of $[kl]$ and l the *second* vertex. (So the first vertex of E_p equals the second vertex of $-E_p$ and vice versa.)

Given any two vertices, r and s, a *path* from r to s is a sequence of edges $e_1 \ldots e_q$ where the first vertex of e_1 is r, the second vertex of e_q is s and the second vertex of e_l equals the first vertex of e_{l+1} for $1 \leq l \leq q-1$. If E_p and $-E_p$ are two consecutive edges in a path we can obtain a shorter path by omitting these two edges. We will call this a *contraction* of the path. It is clear that given any two vertices, r and s, there is a unique shortest path from r to s and that this path can be obtained from any path that goes from r to s by a sequence of contractions. We denote this shortest path by $[r, s]$. It should be noted that we cannot assign an orientation to the path in general, but each of the edges that makes up this path does have an orientation.

Given a path, P, from r to s and any edge, E_j, we will let $\Sigma_P E_j$ denote the number of times that E_j appears in the path minus the number of times that $-E_j$ appears in the path. The following lemma is immediate.

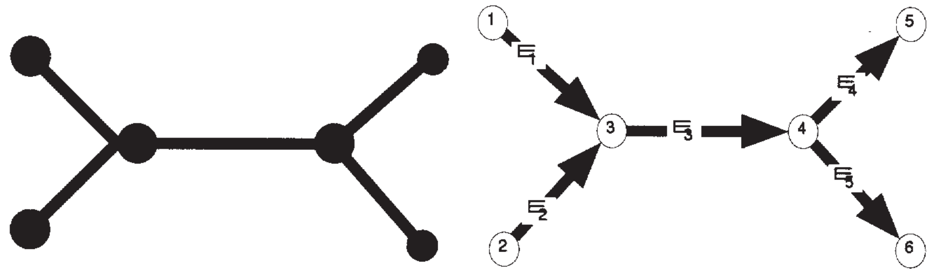

FIGURE 1. Tree with a labeling of edges and vertices

Lemma 1. *Let T be a tree with r and s as vertices. Let P denote any path from r to s . Then $\Sigma_P E_j = \Sigma_{[r,s]} E_j \in \{0, 1, -1\}$.*

Example 1. *Figure 1 shows a tree and a labeling of the vertices and edges. A path from vertex 2 to vertex 6 is given by $E_2{-}E_1E_1E_3E_4{-}E_4E_5$. The shortest path is $E_2E_3E_5 = [2,6]$.*

Suppose θ is a permutation on $1,\ldots,n$. Then we define L_θ to be the continuous map from T to itself given by $L_\theta([k,l]) = [\theta(k),\theta(l)]$ for each edge $[k,l]$ in the tree. (It should be noted that this map depends on the labeling of the vertices of the tree.)

Associated to each permutation $\theta \in S_n$ is a directed graph called *the Markov graph* of the permutation and denoted by $G(\theta)$. It is defined as follows: there are $n-1$ vertices denoted by $E_1,\ldots,E_{n-1}$, and there is a positive arrow from E_j to E_k if and only if E_k appears in the path $L_\theta(E_j)$ and a negative arrow from E_j to E_k if and only if $-E_k$ appears in the path $L_\theta(E_j)$.

(We are abusing the notation and letting E_r denote both the vertex in the graph and an edge of the tree. This should cause no confusion as the context should make the meaning of E_r clear.)

The following lemma is a basic result that will be used repeatedly throughout the rest of this paper. It is an easy consequence of the intermediate value theorem (see [9] or [5], for example, for a formal proof).

Lemma 2. *Let $\theta \in S_n$. Suppose that $E_{k_0} \to E_{k_1} \to \ldots E_{k_m} \to E_{k_0}$ is a loop in $G(\theta)$. Then there exists a periodic point x of L_θ with $L_\theta^{m+1}(x) = x$ such that $L_\theta^r(x) \in E_{k_r}$ for $r = 0,1\ldots,m$. Conversely, if x is a periodic point in L_θ of period $m+1$ and if x is not a vertex then there exists a loop $E_{k_0} \to E_{k_1} \to \ldots E_{k_m} \to E_{k_0}$ in $G(\theta)$ such that $L_\theta^r(x) \in E_{k_r}$ for $r = 0,1\ldots,m$.*

We can extend the first half of this lemma to continuous maps.

Lemma 3. *Suppose that f is a continuous map on a tree with n vertices labeled 1 through n. Let $\theta \in S_n$ be such that $f(i) = \theta(i)$ for each vertex. Suppose that $E_{k_0} \to E_{k_1} \to \ldots E_{k_m} \to E_{k_0}$ is a loop in $G(\theta)$. Then there exists a periodic point x of f with $f^{m+1}(x) = x$ such that $f^r(x) \in E_{k_r}$ for $r = 0,1\ldots,m$.*

Proof. This result follows immediately from the previous lemma after noting that if $E_{k_i} \to E_{k_{i+1}}$ is an arrow in the Markov graph of θ then there is a closed subinterval I contained in E_{k_i} with $f(I) = E_{k_{i+1}}$. □

If θ consists of a single cycle and if the loop $E_{k_0} \to E_{k_1} \to \ldots E_{k_m} \to E_{k_0}$ in the above statement is not a repetition of a shorter loop then it can be checked that

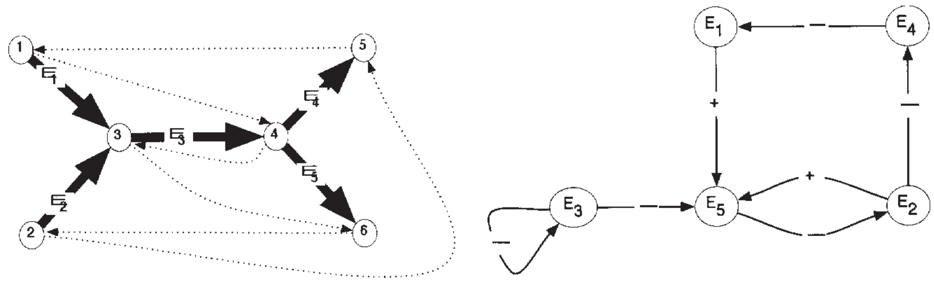

FIGURE 2. Tree with vertex permutation (143625) and associated Markov graph.

the period of x is $m+1$. In such a case we will say the that there is a *non-repetitive* loop of length $m+1$.

To prove the major theorem of the paper it is enough to prove the following:

Lemma 4. *Let T be a tree and f a continuous map from T to itself. Suppose that the tree has n vertices labeled 1 through n. Let $\theta \in S_n$ be such that $f(i) = \theta(i)$ for each vertex. Suppose that θ consists of a single cycle of length n.*

1) If n is not a divisor of 2^k then $G(\theta)$ has a non-repetitive loop of length 2^k.

2) If $n = 2^k m$, where $m > 1$ is odd and $k \geq 0$. Then $G(\theta)$ has a non-repetitive loop of length $2^k l$ for any $l \geq m$.

3) If n is odd then $G(\theta)$ has a non-repetitive loop of length $n-1$.

We will prove this theorem in section 4, but first we need to introduce the orientation of loops and the associated transition matrix. A loop in a Markov graph has *positive orientation* if and only an even number of the arrows that comprise the loop are negative, and *negative orientation* if and only an odd number of the arrows that comprise the loop are negative. The following lemma, though elementary, is important.

Lemma 5. *The orientation of a loop is independent of the choice of the orientations of the edges.*

Proof. It is trivial to check if the loop is of length 1. Suppose that E_i is a vertex in a loop of length greater than 1. Changing the orientation of the edge E_i changes the sign of the arrow entering E_i and leaving E_i and thus has no effect on the orientation of the loop. □

Assigned to each $\theta \in S_n$ is an *oriented transition matrix*, denoted $M(\theta)$ and defined by $M(\theta)_{i,j} = \begin{cases} 1 & \text{if there is a positive arrow from } E_j \text{ to } E_i, \\ -1 & \text{if there is a negative arrow from } E_j \text{ to } E_i, \\ 0 & \text{otherwise.} \end{cases}$

The proof of the following lemma is elementary and left to the reader.

Lemma 6. *Let $\theta \in S_n$. Then the ijth entry of $[M(\theta)]^k$ equals the number of orientation preserving paths from I_i to I_j of length k minus the number of orientation reversing paths of length k.*

Example 2. *Figure 2 shows a tree with the vertices permuted by* (143625) *and the associated Markov graph. It is clear from the graph that there are non-repetitive*

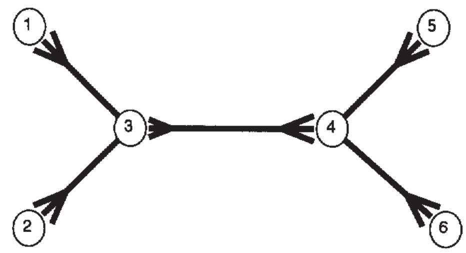

FIGURE 3. Tree with vertex permutation (143625) showing arrowheads.

loops of all even periods. Apart from loops of even lengths there is one loop of length 1.

The oriented transition matrix associated to this tree, permutation and labeling is $M(\theta) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix}$. *Notice that* $[M(\theta)]^6$ *is the identity matrix, so that for each vertex the number of orientation preserving loops of length* 6 *minus the number of orientation reversing loops of length* 6 *is* 1. *We will show later that if* n *is an even number not divisible by* 6 *then the total number of orientation preserving loops of length* n *minus the number of orientation reversing loops of length* n *is* -1.

We conclude this section with a result that will be used repeatedly in the proof of the major theorem.

Theorem 2. *Let* T *be a tree. Suppose that the tree has* n *vertices labeled* 1 *through* n. *Let* $\theta \in S_n$. *If* $\theta(i) \neq i$ *for each vertex* i *then* $Trace(M(\theta)) = -1$.

Proof. For each vertex, i, consider the path $[i, \theta(i)]$. On the first edge of this path draw an arrowhead by vertex i, but pointing away from it. (This can be thought of as the direction in which the vertex i moves under θ.) Observe that if edge E_i has two arrowheads on it then it covers itself with a negative orientation and so there is a negatively oriented arrow from E_i to itself in $M(\theta)$; if there are no arrowheads on E_i then it covers itself with a positive orientation and so there is a positively oriented arrow from E_i to itself in $M(\theta)$; and that if there is exactly one arrowhead on E_i then it doesn't cover itself and so there is no arrow from E_i to itself.

Let $A(i)$ denote the number of arrowheads on E_i. Then the ith entry along the main diagonal of $M(\theta)$ is $1-A(i)$. So $Trace(M(\theta))$ is $\sum_1^{n-1}(1-A(i)) = (n-1)-n = -1$. □

3. **Algebra.** In the previous section it was shown that given a tree with n vertices and a labeling there was a natural way to assign a matrix to each element of S_n. In this section we will show that this assignment gives an irreducible representation of S_n.

Theorem 3. *Let* T *be a tree. Suppose that the tree has* n *vertices labeled* 1 *through* n. *Suppose that* $\alpha, \beta \in S_n$. *Then* $M(\alpha)M(\beta) = M(\alpha\beta)$.

Proof. Let $E_j = [r,s]$ be an edge in the tree. By definition we have $L_\beta(E_j) = L_\beta[r,s] = [\beta(r),\beta(s)]$ and $L_{\alpha\beta}(E_j) = [\alpha(\beta(r)),\alpha(\beta(s))]$. Now $L_\alpha(L_\beta(E_j))$ is a path from $\alpha(\beta(r))$ to $\alpha(\beta(s))$ which we will denote by P.

The edge $E_i \in L_\alpha(L_\beta(E_j))$ if and only there exists a k such that either $E_k \in L_\beta(E_j)$ and $L_\alpha(E_k) = E_i$ or such that $-E_k \in L_\beta(E_j)$ and $L_\alpha(-E_k) = E_i$. This means that $E_i \in L_\alpha(L_\beta(E_j))$ if and only there exists a k such that $M(\alpha)_{i,k}M(\beta)_{k,j} = 1$. Similarly, $-E_i \in L_\alpha(L_\beta(E_j))$ if and only there exists a k such $M(\alpha)_{i,k}M(\beta)_{k,j} = -1$. Thus $M(\alpha)M(\beta))_{i,j} = \Sigma_P E_i$.

By definition $M(\alpha\beta)_{i,j} = \Sigma_{[\alpha(\beta(r)),\alpha(\beta(s))]}E_i$ and Lemma 1 shows that $\Sigma_P E_i = \Sigma_{[\alpha(\beta(r)),\alpha(\beta(s))]}E_i$, so $(M(\alpha)M(\beta))_{i,j} = M(\alpha\beta)_{i,j}$. □

We conclude this section with two theorems that relate the algebraic approach to trees taken in this paper to basic algebraic properties. Theorem 4 shows that we can define trees as minimal generating sets of transpositions, and Theorem 5 shows that the representations of S_n given by our construction are irreducible. These results are included for completeness, but it should be noted that neither will be used in the following sections.

Let T be a tree with n vertices labeled 1 through n. If $[r,s]$ is an edge we will call the transposition $(rs) \in S_n$ an *edge transposition.* We will say a set S of transpositions is a *minimal generating set for* S_n if every element of S_n can be generated by products of these transpositions and if no proper subset of S generates S_n.

Theorem 4. *Let T be a tree with n vertices labeled* 1 *through n. Then the set of edge transpositions is a minimal generating set for S_n. Conversely, given a minimal generating set of transpositions for S_n the associated graph is a tree.*

Proof. We will prove the first part and leave the proof of the converse to the reader. First we will show that if there is a path from a to b in the tree then the transposition (ab) can be generated. We do this by induction on the length of the path. Clearly, if there is a path from a to b that has just one edge then (ab) is an edge transposition. Suppose that we have shown that given any path, $[c,d]$, of length $k-1$ that (cd) can be generated. Given a path $[f,g]$ of length k. Let $[h,g]$ denote the last edge in the path. Then by the inductive hypothesis we know (fh) can be generated. We also know that (hg) is an edge transposition and that $(hg)(fh)(hg) = (fg)$, so (fg) can be generated.

Since there is a path from any vertex to any other vertex we can generate all the transpositions in S_n and these certainly generate all of S_n.

We now show that the set of edge transpositions is minimal. Let S denote the set of edge transpositions. Suppose that (rs) is an edge transposition and consider the set $U = S - \{(rs)\}$. We will show that U cannot generate S_n. We consider two cases depending on whether $[r,s]$ is a leaf in the tree. Suppose that $[r,s]$ is a leaf and without loss of generality assume that the vertex r belongs just to this edge. Then none of the other edge transpositions contains the number r and so U clearly cannot generate (rs). If $[r,s]$ is not a leaf then consider T with this edge deleted. Clearly $T - [r,s] = T_1 \cup T_2$ where T_1 and T_2 are disjoint trees. The set U can be written as $U_1 \cup U_2$ where $(ab) \in U_i$ if and only if $[a,b] \in T_i$. Then U_1 generates the symmetric group on the nodes in T_1, call this group S_{T_1}, and U_2 generates the symmetric group on nodes in T_2, call this group S_{T_2}. Then U generates exactly $S_{T_1} \times S_{T_2}$, but $S_{T_1} \times S_{T_2} \not\cong S_n$. □

Theorem 3 shows that the assignment of matrices to permutations on trees gives rise to representations of S_n. The following theorem shows that these representations are irreducible. (In [8] this was shown for the special case when the tree is homeomorphic to the interval.)

Theorem 5. *Let T be a tree. Suppose that the tree has n vertices labeled 1 through n. Then $M : S_n \to GL(n-1, R)$ is an irreducible representation of S_n.*

Proof. Theorem 3 shows that M gives a representation. We need to show that this representation is irreducible. To prove this we must show that for any non-zero vector $\mathbf{u} \in \mathbf{R}^{n-1}$ the span of the set of vectors $\{M(\sigma)\mathbf{u}|\sigma \in S_n\}$ is $\mathbf{R}^{n-1}$. To do this we introduce the standard basis for $\mathbf{R}^{n-1}$, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}\}$, and show that for any of these vectors, $\mathbf{e}_i$, that $Span(M(\sigma)\mathbf{e}_i|\sigma \in S_n)$ is $\mathbf{R}^{n-1}$, then we show that for any non-zero vector $\mathbf{u}$ there is at least one of these basis vectors in $Span(M(\sigma)\mathbf{u}|\sigma \in S_n)$.

We restrict attention to edge transpositions in S_n. Let $E_i = [r, s]$ be an edge. Then it is easy to check that $M((rs)) - I$ has zeroes everywhere except in the i-th row. The j-th entry of the i-th row is non-zero if and only if E_j abuts E_i. In this case we have $(M((rs)) - I)\mathbf{e}_j = c\mathbf{e}_i$ for some non-zero integer c. So if E_j and E_i abut we have $\mathbf{e}_i \in Span(M(\sigma)\mathbf{e}_j|\sigma \in S_n)$ and $\mathbf{e}_j \in Span(M(\sigma)\mathbf{e}_i|\sigma \in S_n)$. Proceeding inductively we can show that if there is a path from E_k to then E_i then $\mathbf{e}_k \in Span(M(\sigma)\mathbf{e}_i|\sigma \in S_n)$. Since we have a tree there is a path from every edge to every other edge and so for each k we have $Span\{M(\sigma)\mathbf{e}_k|\sigma \in S_n\}$ is $\mathbf{R}^{n-1}$

Let the k-th entry of $\mathbf{u}$ be non-zero. Let $E_k = [p, q]$. Then $(M(pq) - I)\mathbf{u}$ is a non-zero scalar multiple of $\mathbf{e}_k$ and so $Span\{M(\sigma)\mathbf{u}|\sigma \in S_n\}$ is $\mathbf{R}^{n-1}$ □

4. **Proof of Theorem** 1. In this section we will show how Lemma 4, and thus Theorem 1, follows from Theorems 2 and 3.

Lemma 7. *Let T be a tree and f a continuous map from T to itself. Suppose that the tree has n vertices labeled 1 through n. Let $\theta \in S_n$ be such that $f(i) = \theta(i)$ for each vertex. Suppose that θ consists of a single cycle of length n. If n is not a divisor of 2^k then $G(\theta)$ has a non-repetitive loop of length 2^k.*

Proof. Since n is not a divisor of 2^k we know that θ^{2^k} contains no $1-cycles$. Theorem 2 shows that $Trace(M(\theta^{2^k}))$ is -1. This means that there is at least one -1 on the main diagonal of $(M(\theta^{2^k})$. Now Theorem 3 tells us that $(M(\theta^{2^k}){=}(M(\theta))^{2^k}$. Then Lemma 6 shows that $G(\theta)$ has a loop of length 2^k with negative orientation. Since the orientation is negative it cannot be the repetition of a shorter loop as any shorter loop would have to be repeated an even number of times. □

Lemma 8. *Let T be a tree and f a continuous map from T to itself. Suppose that the tree has n vertices labeled 1 through n. Let $\theta \in S_n$ be such that $f(i) = \theta(i)$ for each vertex. Suppose that θ consists of a single cycle of length n. If $n = 2^k p$, where $p > 1$ is odd and $k \geq 0$ then $G(\theta)$ has a non-repetitive loop of length $2^k m$ for any $m \geq p$.*

Proof. Theorem 2 tells us that $Trace(M(\theta^{2^k}))$ is -1. So by Theorem 3 and Lemma 6 there is a loop of length 2^k with negative orientation in $G(\theta)$. Since $\theta^{2^k p}$ is the identity, we have $M(\theta^{2^k p}) = I$. So, using Theorem 3 and Lemma 6, we know that there is a loop of length $2^k p$ with positive orientation for each of the vertices in $G(\theta)$. Thus one vertex has a loop of length 2^k and a loop of length $2^k p$. Since

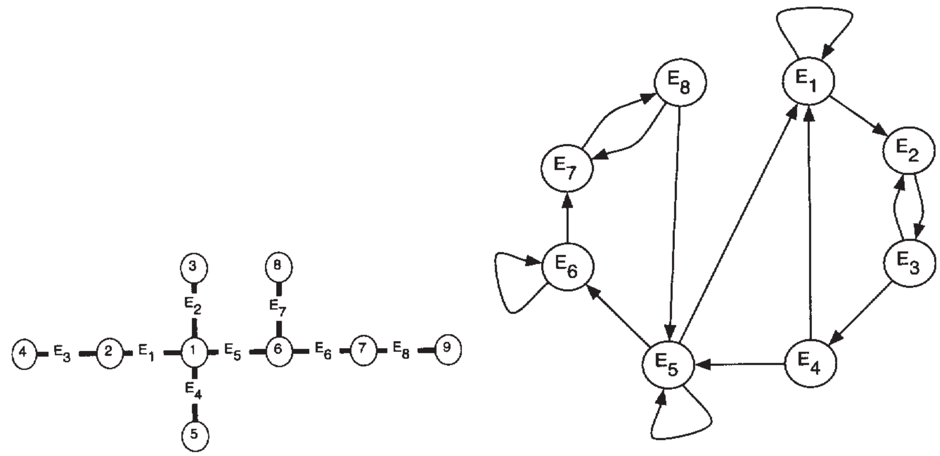

FIGURE 4. Tree with vertex permutation (123456789) and associated Markov graph.

the loop of length $2^k p$ has positive orientation, the loop of length 2^k has negative orientation, and p is odd, it follows that the longer loop is not a repetition of the shorter one. Thus we can form a non-repeating loop of length $2^k m$, where $m \geq p$ by first going around the loop of length $2^k p$ once and then going around the loop of length 2^k $m - p$ times. $\square$

The above lemma shows that if n is odd then $G(\theta)$ will have a non-repetitive loop of length m for any $m > n$. However, as the following lemma shows we can strengthen this result slightly.

Lemma 9. *Let T be a tree and f a continuous map from T to itself. Suppose that the tree has n vertices labeled* 1 *through n. Let $\theta \in S_n$ be such that $f(i) = \theta(i)$ for each vertex. Suppose that θ consists of a single cycle of length n. If n is odd then $G(\theta)$ has a non-repetitive loop of length $n - 1$.*

Proof. As in the proof of the previous lemma we know that there is a vertex that has a loop of length 1 and a loop of length n that is not a repetition of the shorter loop. However, $G(\theta)$ only has $n - 1$ vertices. Thus there must be a loop of length less than n from this vertex that is not a repetition of the loop of length 1. Going around this loop once and then around the 1-loop the appropriate number of times gives a non-repetitive loop of length $n - 1$. $\square$

For maps of the interval it can be shown that if L_α has more than one fixed point then L_α has periodic points of all periods (see [7] or [9]). This is the result that is needed to complete the proof of Sharkovsky's Theorem. However, for maps of trees this result is no longer necessarily true as the example in figure 4 shows. In this example the Markov graph shows that there are three fixed points, but there is no loop of length three and so no periodic orbits of period 3.

5. **A family of trees and maps.** We will now define a family of trees, T_n, recursively on the number of vertices n. If $n = 1$ there is just one possible tree and labeling, a single vertex labeled 1. Suppose that the tree and labeling has been defined for 2^m vertices. For k satisfying $2^m < k \leq 2^{m+1}$ define a new edge that goes from vertex k to vertex $k - 2^m$.

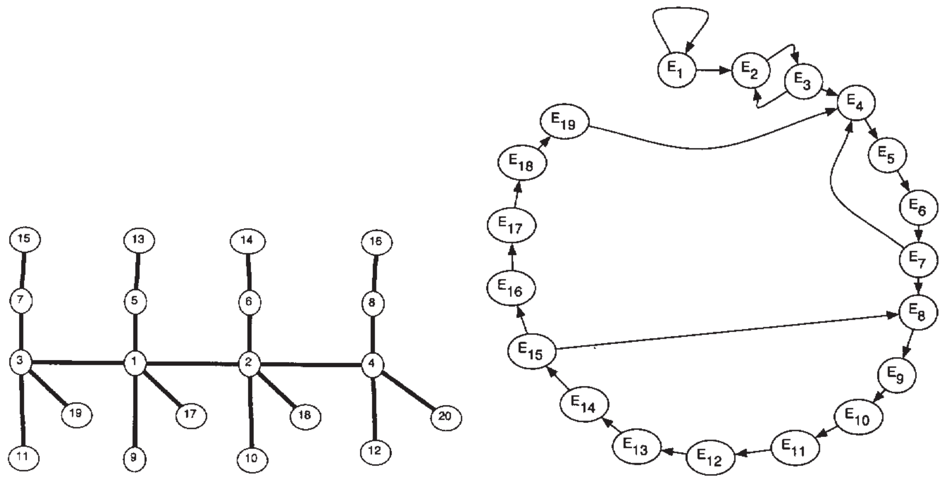

FIGURE 5. T_{20} and associated Markov graph.

Given the tree T_n we choose the map to be L_θ where θ is the permutation $(123\ldots n)$. (Figure 5 shows T_{20} and its associated Markov graph.) We will now classify the sets of periods that T_n can have.

Theorem 6. *Let T_n be as defined above.*

If $n = 2^m$ then T_n has periodic points with periods 2^k for each k satisfying $2^k \leq n$ and there are no others.

If $n = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_j}$ where $r_i < r_{i+1}$ and $j > 1$ then T_n has periodic points with periods 2^k for each k, periods $n + v2^{r_1}$ for any v, periods $\sum_{t=m}^{j} 2^{r_t}$ for $m \geq 1$, and there are no others.

Proof. Label the edges in the following way. If $[a, b]$ is an edge in the tree with $a < b$ denote this edge E_{b-1}. We consider the Markov graph associated to the map. It is straightforward to check that for $i < n - 1$ there is an arrow from E_i to E_{i+1} and that if $i = 2^k - 1$ then there is an arrow from E_i to $E_{2^{k-1}}$ and that these are the only arrows leaving E_i. This means that if we exclude the last vertex, E_{n-1}, the Markov graph associated to the tree consists of a sequence of loops of length 2^k, where $2^k < n - 1$. These loops can be written as $E_{2^k} \to E_{2^k+1} \to E_{2^k+2} \to \ldots E_{2^{k+1}-1} \to E_{2^k}$. For each of these loops of length 2^k, with the exception of the loop of length 1, there is one, and only one, arrow coming into it. This arrow comes from the loop of length of length 2^{k-1}. For each of these loops there is only one arrow leaving it. This is the arrow from $E_{2^{k+1}-1}$ to $E_{2^{k+1}}$. Thus if we ignore the vertex E_{n-1}, these loops of length 2^k for $2^k < n - 1$ are the only loops.

Suppose that $n = 2^m$. Then $E_{n-1} = [2^{m-1}, 2^m]$. Since $L_\theta(2^m) = 1$ and $L_\theta(2^{m-1}) = 2^{m-1} + 1$ and $E_{2^{m-1}} = [1, 2^{m-1} + 1]$, there is an arrow from E_{n-1} to $E_{2^{m-1}}$, and so the associated Markov graph has loops of of length 2^k for each k satisfying $2^k < n$ and these are the only loops. Thus L_θ has only periodic points of period 2^k for each k satisfying $2^k \leq n$.

Suppose $n = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_j}$ where $r_i < r_{i+1}$. Then $E_{n-1} = [2^{r_1} + 2^{r_2} + \cdots + 2^{r_{j-1}}, n]$. Since $L_\theta(n) = 1$ and $L_\theta(2^{r_1} + 2^{r_2} + \cdots + 2^{r_{j-1}}) = 1 + 2^{r_1} + 2^{r_2} + \cdots + 2^{r_{j-1}}$ we have $L_\theta(E_{n-1}) = [1, 1 + 2^{r_1} + 2^{r_2} + \cdots + 2^{r_{j-1}}]$. Now $[1, 1 + 2^{r_1} + 2^{r_2} + \cdots + 2^{r_{j-1}}] = [1, 1 + 2^{r_1}] \cup [1 + 2^{r_1}, 1 + 2^{r_1} + 2^{r_2}] \cup \cdots \cup [1 + 2^{r_1} + \cdots + 2^{r_{j-2}}, 1 + 2^{r_1} + \cdots + 2^{j-1}] =$

$E_{2^{r_1}} \cup E_{2^{r_1}+2^{r_2}} \cup \cdots \cup E_{2^{r_1}+\ldots+2^{r_{j-1}}}$. Thus the Markov graph will have arrows from E_{n-1} to $E_{2^{r_1}}, E_{2^{r_1}+2^{r_2}}, \ldots, E_{2^{r_1}+\ldots+2^{r_{j-1}}}$.

Now consider the arrow from E_{n-1} to $E_{\sum_{t=1}^{m} 2^{r_t}}$ for $1 \leq m \leq j-1$. This adds the loop $E_{n-1} \to E_{\sum_{t=1}^{m} 2^{r_t}} \to E_{\sum_{t=1}^{m} 2^{r_t}+1} \to \ldots E_{n-2} \to E_{n-1}$. This loop has length $\sum_{t=1}^{j} 2^{r_t} - \sum_{t=1}^{m} 2^{r_t} = \sum_{t=m+1}^{j} 2^{r_t}$. It intersects the loops of length 2^k for $k \geq m$, in addition to the other loops that pass through E_{n-1}. Thus the lengths of only non-repetitive loops of length less than n that can be constructed using this loop of length are $\sum_{t=m+1}^{j} 2^{r_t}$ and $\sum_{t=m}^{j} 2^{r_t}$. All the lengths of non-repetitive paths of lengths greater than n have the form $n + v2^{r_1}$ for some v.

□

We conclude by noting that sets of periods given in Theorem 1 coincide with the sets of periods in Theorem 6 when the binary expansion of n contains two or less 1s. So in these cases Theorem 1 is best possible.

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E-mail address: cbernhardt@mail.fairfield.edu